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Crack tip reinforcement by bridging elements: modelling the fracture of the matrix material

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There is a wide range of physical situations where the faces of a crack in a matrix material with limited ductility are restrained from opening, a phenomenon known as crack reinforcement. In quantifying this phenomenon, the simplest way of modelling the behaviour of the matrix material ahead of a crack tip is to assume that it deforms in accord with the laws of linear elasticity with the crack extending when the stress intensity at the crack tip (the leading edge of the restraining region) attains a critical value, K_{IC}, the fracture toughness of the matrix material; i.e. the details of the matrix deformation and fracture behaviour are ignored. The viability of this K-matrix assumption is examined for the case of a semi-infinite crack in a remotely loaded infinite solid, for which the restraining stress between the crack faces increases linearly with crack opening until the attainment of a critical opening when the restraining stress falls to zero. The analysis defines the range of material parameters for which the K-matrix assumption is adequate with regard to the determination of (a) the "applied" value of K required for the crack tip and restraining zone to propagate through the solid, and (b) the size of the restraining zone when propagation occurs. The K-matrix assumption always gives an overestimate of the applied K, but the overestimation is small when the matrix toughness contribution is small or is dominant. However, when the contributions from the matrix toughness and the toughness provided by the restraining material are roughly equivalent, the K-matrix assumption leads to a significant overestimate of the applied K, and in this situation the matrix material behaviour should be modelled more precisely.

1. Introduction

There is a wide range of technological problems in materials science and engineering where a solid contains a crack and the crack faces are restrained from opening; this leads to a so-called reinforcement of the crack tip. Some of these situations have been identified by Rose [1]: fibre bridging of cracks in composites, restraint due to unbroken ligaments during cleavage cracking, rubber-inclusion toughening of structural adhesives, craze formation in glassy polymers, aggregate interlocking in concrete, the repair of cracked plates by bonded reinforcements, and part-through cracking in laminated plates. These various situations, which embrace both microstructural and macro levels, can be simulated by a model in which the faces of a planar crack are restrained by forces whose average behaviour can be represented by an appropriate stress–crack opening relationship. Two recent papers [1, 2] have addressed this general problem from very different perspectives, in that Rose [1] examined the situation where the restraining stress increases (linearly) with the crack opening, whereas Foote *et al.* [2] examined the situation where the stress decreases with crack opening, referring to this phenomenon as strain-softening. The results from these two approaches, taken together, should be applicable to most of the examples referred to in this paragraph.

With both approaches, it has been assumed that the material immediately ahead of the crack tip, i.e. the leading edge of the restraining zone, deforms in accord with the laws of linear elasticity, with the crack extending when the stress intensity at the crack tip attains a critical value, K_{IC}, the fracture toughness of the matrix material (Mode I crack opening is assumed); i.e. the details of the matrix deformation and fracture behaviour are ignored (hereafter, this is referred to as the K-matrix assumption). Coupling this assumption with the additional assumption that the restraining material loses its cohesion when the crack opening attains a critical value, ϕ_m, allows one to determine the critical value of the "applied" stress intensity that is required for the crack tip together with its restraining zone to propagate through the solid. An obvious question arises as to whether, and under what conditions, the K-matrix assumption is valid. The present paper addresses this question for the case of a two-dimensional semi-infinite crack in a remotely loaded infinite solid (Mode I deformation), where the restraining stress between the crack faces increases linearly with the crack opening, in terms of the accuracy of: (a) the determined applied K value, and (b) the calculated size of the restraining zone, a parameter which is important when making correlations with experimental crack tip observations.

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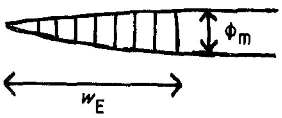

Figure 1 The mode I model of a semi-infinite crack in a remotely loaded infinite elastic solid. The restraining zone size is w_E.

2. The applied stress intensity

Fig. 1 shows the Mode I plane strain model of a semi-infinite crack in a remotely loaded infinite solid; the solid and particularly the material ahead of the crack tip, deforms in accord with the laws of linear elasticity. The crack faces are restrained from opening by stresses such that the restraining stress–crack opening law is linear (Fig. 2), with the restraint suddenly falling to zero when the crack opening, ϕ, attains a critical value, ϕ_m.

Straightforward application of the J-integral approach [3] shows that the applied K value, i.e. K_E, for the crack tip together with its restraining zone to propagate through the solid is given by the expression

$$K_E^2 = K_{IC}^2 + \frac{E}{(1-\nu^2)}\int_0^{\phi_m}\sigma(\phi)\mathrm{d}\phi$$

$$= K_{IC}^2 + \frac{E}{(1-\nu^2)}\frac{\sigma_m\phi_m}{2} \qquad (1)$$

where K_{IC} is the fracture toughness of the matrix material; E and ν are, respectively, the Young's modulus and Poisson's ratio of the matrix material.

Now if the matrix material ahead of the crack tip has a limited ductility, i.e. it is able to deform plastically albeit with difficulty, the simplest way of modelling this deformation is to confine it to an infinitesimally thin zone (Fig. 3) in which there is a constant cohesive stress, σ_p, such that there is no singularity at the leading edge of this zone. It is also presumed that the matrix material fractures (at the crack tip) when the relative displacement within this zone attains a critical value, ϕ_p (see Figs 3 and 4). In this case, the crack tip opening is ϕ_p and not zero as is the case with the K-matrix assumption. For this new situation it immediately follows, again by application of the J-integral approach, that the critical K value, i.e. K_p, for the crack tip together with its restraining zone to propagate through the solid is given by the expression

$$K_p^2 = \frac{E\sigma_p\phi_p}{(1-\nu^2)} \qquad \text{if } \phi_m < \phi_p \qquad (2)$$

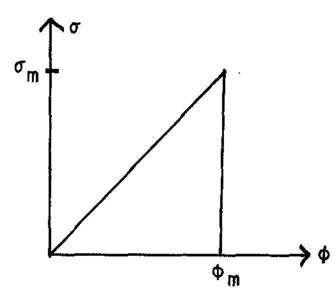

Figure 2 The stress–crack opening law for the restraining material.

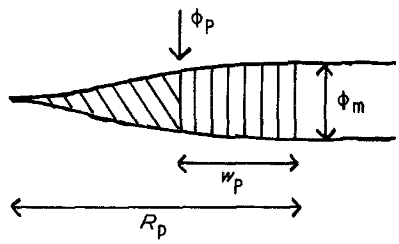

Figure 3 The Mode I model of a semi-infinite crack in a remotely loaded infinite solid; there is a plastic zone ahead of the crack tip. The restraining zone size is w_p and the plastic zone size is $R_p - w_p$.

and

$$K_p^2 = \frac{E}{(1-\nu^2)}\left[\sigma_p\phi_p + \frac{\sigma_m\phi_m}{2}\left(1 - \frac{\phi_p}{\phi_m}\right)^2\right]$$
$$\text{if } \phi_m > \phi_p \qquad (3)$$

These results (Equations 2 and 3), which are based on the assumption that the matrix can deform plastically, may be compared with that (Equation 1) obtained via the K-matrix assumption, by inputting the K_{IC} value, i.e. $[E\sigma_p\phi_p/(1-\nu^2)]^{1/2}$, of the matrix material into Equation 1

$$K_E^2 = \frac{E}{(1-\nu^2)}\left[\sigma_p\phi_p + \frac{\sigma_m\phi_m}{2}\right] \qquad \text{all } \phi_m, \phi_p \qquad (4)$$

It then follows from Equations 2, 3 and 4 that

$$\frac{K_E^2}{K_p^2} = 1 + \frac{\sigma_m\phi_m}{2\sigma_p\phi_p} \qquad \text{if } \phi_m < \phi_p \qquad (5)$$

$$\frac{K_E^2}{K_P^2} = \frac{1 + (\sigma_m\phi_m/2\sigma_p\phi_p)}{1 + (\sigma_m\phi_m/2\sigma_p\phi_p)\,[1 - (\phi_p/\phi_m)]^2}$$
$$\text{if } \phi_m > \phi_p \qquad (6)$$

and it is immediately seen that K_E always exceeds K_p. Values of K_E/K_p for the cases $\phi_m/\phi_p = 0.5$, 1 and 2, and for values of σ_m/σ_p ranging between 0.25 and 2.00 are shown in Table I. These results show that K_E can differ appreciably from K_p, and that within the ranges of values ϕ_m/ϕ_p and σ_m/σ_p for which results are given, the difference between K_E and K_p increases with ϕ_m/ϕ_p and σ_m/σ_p. However, inspection of Equation 6 shows that when ϕ_m/ϕ_p is very large, then $K_E \sim K_p$.

Broadly speaking, therefore, the K-matrix assumption always leads to an overestimate of the crack tip stress intensity required to propagate the crack tip and its restraining zone through the solid. The degree of overestimation is small for the two extremes where the

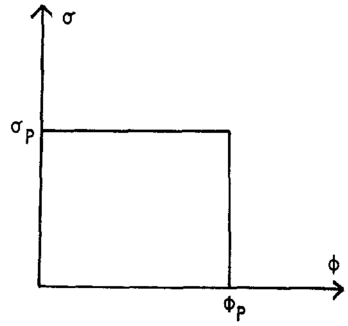

Figure 4 The stress–relative displacement relation for the material in the plastic zone.

TABLE I The ratio K_E/K_p for different values of ϕ_m/ϕ_p and σ_m/σ_p

σ_m/σ_p	K_E/K_p		
	$\phi_m/\phi_p = 0.5$	$\phi_m/\phi_p = 1.0$	$\phi_m/\phi_p = 2.0$
0.25	1.030	1.061	1.085
0.50	1.061	1.118	1.155
0.75	1.089	1.173	1.214
1.00	1.118	1.225	1.265
1.25	1.145	1.275	1.310
1.50	1.173	1.323	1.348
1.75	1.199	1.369	1.383
2.00	1.225	1.414	1.414

contribution of the matrix toughness to the overall toughness is both small and large, with the greatest degree of overestimation arising when the contribution from the matrix toughness and the toughness provided by the restraining material are roughly equivalent.

3. The size of the restraining zone

As indicated in Section 1, a knowledge of the restraining zone size is important when making correlations with experimental crack tip observations. Unlike the crack tip stress intensity required to propagate the crack tip and its restraining zone through the solid, where is is easy to obtain an analytical result (see Section 2), this is not possible when determining the restraining zone size. Instead of using a numerical approach, we will assume that the stress retains a constant value, σ_a, within the restraining zone, a situation for which analytical results are readily obtained. If σ_a is set equal to $\sigma_m/2$, the results for the constant stress case should provide a reasonable guide to the behaviour of restraining material with a linear hardening law.

If the matrix material is assumed to deform elastically (i.e. the K-matrix assumption), because the restraining crack tip stress intensity is $(2/\pi)\sigma_a(2\pi w_E)^{1/2}$ [4], where w_E is the size of the restraining zone, matching of the stress intensities at the crack tip gives the expression

$$K_E = K_{IC} + \frac{2\sigma_a}{\pi}(2\pi w_E)^{1/2} \qquad (7)$$

where K_E is the "applied" crack tip stress intensity needed to propagate the crack tip together with its restraining zone through the solid. By analogy with Equation 1, for the constant stress restraining law, K_E is also given by the equation

$$K_E^2 = K_{IC}^2 + \frac{E}{(1-\nu^2)}\sigma_a\phi_m \qquad (8)$$

where ϕ_m is again the crack opening at which the restraining stress drops to zero. Thus again inputting the K_{IC} value of the matrix, i.e. $[E\sigma_p\phi_p/(1 - \nu^2)]^{1/2}$, and writing $\sigma_a = q\sigma_p$ and $\phi_m = \lambda\phi_p$, it follows from Equations 7 and 8 that the size, w_E, of the restraining zone is given by the equation

$$\frac{8(1-\nu^2)\sigma_a w_E}{\pi E\phi_m} = \left[\left(1+\frac{1}{\lambda q}\right)^{1/2} - \left(\frac{1}{\lambda q}\right)^{1/2}\right]^2 \qquad (9)$$

On the other hand, if due account is taken of the plastic deformation ahead of the crack tip, as in the analysis in Section 2 for K_p, the cohesive stress is σ_p for $0 < \phi < \phi_p$ and σ_a for $\phi_p < \phi < \phi_m$, whereupon the size w_p of the restraint region (i.e. where $\phi_p < \phi < \phi_m$) is given by the simultaneous equations [5]

$$q\psi\Big/\left[\frac{8(1-\nu^2)\sigma_a w_p}{\pi E\phi_m}\right] = [q + (1-q)(1-\psi)^{1/2}] + \frac{(1-q)\psi}{2}\ln\left[\frac{1+(1-\psi)^{1/2}}{1-(1-\psi)^{1/2}}\right] \qquad (10)$$

$$q\psi\Big/\lambda\left[\frac{8(1-\nu^2)\sigma_a w_p}{\pi E\phi_m}\right] = (1-\psi)^{1/2}[q + (1-q)(1-\psi)^{1/2}] - \frac{q\psi}{2}\ln\left[\frac{1+(1-\psi)^{1/2}}{1-(1-\psi)^{1/2}}\right] \qquad (11)$$

with $\psi = w_p/R_p$, and R_p is the combined size of the plastic and restraint regions, i.e. where $0 < \phi < \phi_m$. The above expressions are valid provided that $\phi_m > \phi_p$, i.e. provided that $\lambda > 1$; if $\phi_m < \phi_p$, i.e. $\lambda < 1$, w_p is of course equal to zero. Equations 10 and 11 can be used to give w_p for prescribed values of q and λ by elimination of ψ. To illustrate the differences between w_p and w_E, consider the special situation where $\sigma_a = \sigma_p$, i.e. $q = 1$, when Equation 9 reduces to

$$\left[\frac{8(1-\nu^2)\sigma_a w_E}{\pi E\phi_m}\right] = \left[\left(1+\frac{1}{\lambda}\right)^{1/2} - \left(\frac{1}{\lambda}\right)^{1/2}\right]^2 \qquad (12)$$

an expression which is valid for all λ, while Equations 10 and 11 simplify to

$$\psi\Big/\left[\frac{8(1-\nu^2)\sigma_a w_p}{\pi E\phi_m}\right] = 1 \qquad (13)$$

$$\psi\Big/\lambda\left[\frac{8(1-\nu^2)\sigma_a w_p}{\pi E\phi_m}\right] = (1-\psi)^{1/2} - \frac{\psi}{2}\ln\left[\frac{1+(1-\psi)^{1/2}}{1-(1-\psi)^{1/2}}\right] \qquad (14)$$

these being valid for $\lambda > 1$; if $\lambda < 1$, then $w_p = 0$. Results obtained from Equations 12, 13 and 14 are shown in Table II. These results clearly show that w_E can differ appreciably from w_p; w_E can be larger or smaller than w_p depending on the value of ϕ_m/ϕ_p.

TABLE II Values of w_E and w_p for $q = \sigma_a/\sigma_p = 1$, and for different values of $\lambda = \phi_m/\phi_p$

$\psi = \frac{w_p}{R_p}$	$\lambda = \frac{\phi_m}{\phi_p}$	$\frac{8(1-\nu^2)\sigma_a w_E}{\pi E\phi_m}$	$\frac{8(1-\nu^2)\sigma_a w_p}{\pi E\phi_m}$	$\frac{w_E}{w_p}$
0.1	1.304	0.206	0.100	2.06
0.2	1.651	0.239	0.200	1.19
0.3	2.111	0.276	0.300	0.92
0.4	2.763	0.320	0.400	0.80
0.5	3.754	0.371	0.500	0.74

4. Discussion

The present paper has examined the viability of the assumption that the material ahead of the tip of a reinforced crack deforms in accord with the laws of linear elasticity, with the crack extending when the stress intensity at the crack tip attains a critical value K_{IC}, the fracture toughness of the matrix material. The viability of this K-matrix assumption has been examined for the case of a semi-infinite crack in a remotely loaded infinite solid, and where the restraining stress between the crack faces increases linearly with crack opening. The K-matrix assumption has been tested with regard to the value of the applied K required for propagation of the crack tip and restraining zone, and also the size of the restraining zone. The results in Tables I and II clearly show that the K-matrix assumption can lead to misleading conclusions as regards the values of these parameters.

In particular the applied crack tip stress intensity is overestimated, the extent of this overestimation being greatest when the contributions from the matrix toughness and the toughness provided by the restraining material are roughly equivalent; when the matrix toughness contribution is small or is dominant, the overestimation is small. Consequently caution should be exercised when modelling the response of the material ahead of the crack tip. The difficulties arise because if this response is assumed to be elastic, there is no crack opening at the crack tip itself; on the other hand, if the matrix material is able to deform plastically, albeit with difficulty, there is a finite crack tip opening. The effects of this opening have been examined in this paper for the case where the restraining stress between the crack faces increases linearly with crack opening, but similar conclusions are expected for more general restraining stress-crack opening laws, where the restraining stress increases with crack opening.

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